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VACUUM VERTICES AND THE GHOST-DILATON

Sabbir Rahman[⋆] and Barton Zwiebach[†]

Center for Theoretical Physics,
Laboratory of Nuclear Science
and Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139, U.S.A.

ABSTRACT

We complete the proof of the ghost-dilaton theorem in string theory by showing that the coupling constant dependence of the vacuum vertices appearing in the closed string action is given correctly by one-point functions of the ghost-dilaton. To prove this at genus one we develop the formalism required to evaluate off-shell amplitudes on tori.

⋆ E-mail address: rahman@mitlns.mit.edu
† E-mail address: zwiebach@irene.mit.edu
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1. Introduction and Summary

Whenever a string background contains a nontrivial ghost-dilaton state, a shift of the string field along this state is expected to alter the value of the dimensionless string coupling constant. This expectation, the so-called "ghost-dilaton theorem", was recently established in Ref.[1] as a property of the covariant closed string field theory action. More precisely, this work established the result only for the field-dependent terms in the string action. In this brief paper we complete the proof of the ghost-dilaton theorem by showing that a shift by the ghost-dilaton also changes the value of the string coupling in the field-independent terms of the string action. These field-independent terms arise as vacuum vertices of nonvanishing genus g.

The effect of the ghost-dilaton shift is found by inserting a ghost-dilaton on the vertices defining the string action. For each vertex, one integrates over the corresponding moduli space of surfaces the result of integrating a ghost-dilaton insertion over each surface. For string vertices *with* external punctures it was found convenient to place the antighost insertions for moduli changes on an external puncture. This choice makes it clear that a suitable ghost-dilaton insertion computes the Euler number of the bordered surfaces comprising the string vertex. Complications arise for the vacuum vertices, though the nature of these difficulties are different for the cases $g \geq 2$ and the case $g = 1$. In general, there are no external punctures available to support the moduli-changing insertions. Moreover, for genus one, the existence of conformal isometries dictates that there is no integral over the position of the dilaton insertion.

In section §2 we demonstrate how the methods of Ref.[1] actually apply for $g \geq 2$ even though the moduli-changing Schiffer vectors are supported on the puncture where the dilaton is inserted. We find that integrating the dilaton over a vacuum vertex correctly reproduces the Euler characteristic of the unpunctured surfaces.

In section §3 we discuss the evaluation of general off-shell amplitudes for punctured tori. This is necessary for the ghost-dilaton, since despite being a physical state it has no primary BRST representative. We construct any once-punctured torus with an arbitrary local coordinate around the puncture by sewing two punctures of a particularly simple three-punctured sphere with an appropriately chosen sewing parameter. We give explicit expressions for the Schiffer vectors which generate the tangents to the moduli space of punctured tori with local coordinates. The results of this section represent an extension to genus one of the techniques developed in Ref.[2].

In section §4 the above ideas are used to show that both the canonical string two-form in the direction of the ghost-dilaton $|D\rangle$ *and* the canonical string one-form in the direction of $|\chi\rangle = -c_0^-|0\rangle$ vanish identically on the moduli space of once-punctured tori (here $|D\rangle = -Q|\chi\rangle$). These results imply that the genus one field-independent terms in the string action are unchanged by a shift of the ghost-dilaton.

2. Vacuum Vertices of Genus $g \geq 2$

In the following we assume some familiarity with the notation of [1], which should be consulted for definitions. If a shift of the ghost-dilaton is to change the coupling constant in the $g \geq 2$ field-independent terms of the string action the following equation should hold (eqn. (8.14) of [1])

$$f_D(\overline{\mathcal{K}}\mathcal{V}_{g,0}) = (2g-2) f(\mathcal{V}_{g,0})\,, \qquad g \geq 2\,. \tag{2.1}$$

The left hand side denotes a ghost-dilaton insertion over the vacuum vertex of genus g, and the right hand side is (minus) the Euler number of a genus g surface times the vacuum vertex. The purpose of the present section is to provide a proof of this equation.

The right hand side involves the integration over $\mathcal{V}_{g,0} \subset \mathcal{M}_{g,0}$, of a $(6g-6)$-form defined in eqn. (3.15) of ref.[3]

$$f(\mathcal{V}_{g,0}) \equiv \mathcal{N}^{3-3g} \int\limits_{\mathcal{V}_{g,0}} d\,\vec{\xi}\; \langle \widehat{\Sigma}\,| b(\widehat{v}_{\xi_1}) \cdots b(\widehat{v}_{\xi_{6g-6}}) |0\rangle\,. \tag{2.2}$$

In this equation $\mathcal{N} \equiv -2\pi i$, and $d\vec{\xi} \equiv d\xi_1 \wedge \cdots \wedge \xi_{6g-6}$, where $(\xi_1, \cdots, \xi_{6g-6})$ is a set of coordinates in $\mathcal{V}_{g,0}$. Given a surface $\Sigma \in \mathcal{V}_{g,0}$, $\widehat{\Sigma} \in \widehat{\mathcal{P}}_{g,1}$ denotes the same surface, with the addition of an extra puncture equipped with a local coordinate (defined up to a phase). In addition, $\widehat{v}_{\xi_i}$ is the Schiffer vector associated to the tangent $\partial/\partial\xi_i$. More precisely, it represents a tangent in $T_{\widehat{\Sigma}}\widehat{\mathcal{P}}_{g,1}$ chosen to project down to $\partial/\partial\xi_i \in T_\Sigma \mathcal{M}_{g,0}$ upon deletion of the extra puncture. While the ingredients used to build the form integrated in (2.2) depend on position and coordinates around the puncture, as well as a choice of representatives for the tangents, the resulting form is independent of this data [4].

Let us elaborate on the ambiguity in the choice of Schiffer vectors representing the tangents in (2.2). If the local coordinate at the puncture is denoted as z, Schiffer vectors $v'(z) = a_0 + a_1 z + \cdots$, regular at the puncture, can only change the position and local coordinates at the puncture. Since the above form is independent of this data, the change $b(\widehat{v}_{\xi_i}) \to b(\widehat{v}_{\xi_i}) + b(v')$ should make no difference in (2.2). This follows immediately from $b(v')|0\rangle = 0$, which, in turn, holds since $b_{n\geq -1}$ and $\bar{b}_{n\geq -1}$ annihilate the vacuum.

Consider now the left hand side of (2.1). Here we must integrate over the position of an extra puncture on each surface in the space $\mathcal{V}_{g,0}$. Since the ghost-dilaton is not primary we need a a family of local coordinates throughout each surface of $\mathcal{V}_{g,0}$. Such a family is obtained by introducing, continuously over $\mathcal{V}_{g,0}$, a conformal metric on every surface. This metric is used to define a family of local coordinates via the prescription of Ref.[5], as elaborated in

§3.1 of [1]. We now write the left hand side of (2.1) as follows

$$f_D(\overline{\mathcal{K}}\mathcal{V}_{g,0}) \equiv \mathcal{N}^{2-3g} \int\limits_{\overline{\mathcal{K}}\mathcal{V}_{g,0}} d\,\vec{\xi} \wedge d\lambda_1 \wedge d\lambda_2 \cdot \langle \widehat{\Sigma}\,|b(v_{\xi_1}) \cdots b(v_{\xi_{6g-6}})\, b(v_{\lambda_1}) b(v_{\lambda_2})|D\rangle\,. \qquad (2.3)$$

The ξ are coordinates $\mathcal{M}_{g,0}$, the same coordinates used in (2.2). The real parameters (λ_1, λ_2) parameterize the position of the dilaton puncture. The Schiffer vectors v_{λ_i} are chosen to alter only the position and coordinate data for the dilaton puncture, while the v_{ξ_i} represent the tangents that change the moduli of the surface, and possibly the data at the puncture. The vectors v_{ξ_i} differ from the vectors $\widehat{v}_{\xi_i}$ appearing in (2.2) by vectors v'_i regular at the puncture. They also may differ by irrelevant "Borel vectors" $\tilde{v}$. Such vectors satisfy $\langle\widehat{\Sigma}|b(\tilde{v}) = 0$.

We now recall from Ref.[1] §5.2 that: $d\lambda_1 \wedge d\lambda_2\, b(v_{\lambda_1}) b(v_{\lambda_2})|D\rangle = iR^{(2)}(\rho)|0\rangle$, where $R^{(2)}(\rho)$ is the curvature two-form associated to the conformal metric ρ used to extract the coordinates for the ghost-dilaton insertion. Replacing this into (2.3) we find

$$f_D(\overline{\mathcal{K}}\mathcal{V}_{g,0}) \equiv -\mathcal{N}^{3-3g} \int\limits_{\mathcal{V}_{g,0}} d\,\vec{\xi} \int\limits_{\Sigma} \cdot \langle \widehat{\Sigma}\,|b(\widehat{v}_{\xi_1}) \cdots b(\widehat{v}_{\xi_{6g-6}})|0\rangle\, \frac{1}{2\pi} R^{(2)}(\rho)\,, \qquad (2.4)$$

where we used the remarks below eqn.(2.2) to replace the vectors v_{ξ_i} by the vectors $\widehat{v}_{\xi_i}$. As emphasized earlier, the form $d\vec{\xi}\,\langle\widehat{\Sigma}\,|b(\widehat{v}_{\xi_1}) \cdots b(\widehat{v}_{\xi_{6g-6}})|0\rangle$ is independent of the position and coordinates of the puncture placed on Σ. Therefore, the integral over Σ in (2.4) can be readily evaluated to give $\frac{1}{2\pi}\int_\Sigma R^{(2)} = 2 - 2g$. Back in (2.4) we find

$$f_D(\overline{\mathcal{K}}\mathcal{V}_{g,0}) = (2g-2)\,\mathcal{N}^{3-3g} \int\limits_{\mathcal{V}_{g,0}} d\,\vec{\xi} \langle \widehat{\Sigma}\,|b(\widehat{v}_{\xi_1}) \cdots b(\widehat{v}_{\xi_{6g-6}})|0\rangle\ . \qquad (2.5)$$

Comparison with (2.2) immediately gives the desired result.

3. Off-Shell Amplitudes in Tori

In this section, we use a representation of the punctured torus in terms of a sewn three-punctured sphere to find the form of the Schiffer vectors which independently generate modulus deformations and local coordinate changes. We then use these to obtain explicit expressions for the canonical forms over the moduli space of tori and in doing so set up for the formalism required to evaluate off-shell amplitudes in tori.

3.1. Once-punctured tori from three punctured spheres

Let $\langle\Sigma_1;w_1|$ and $\langle\Sigma_2;w_2|$ denote the surface states corresponding to the punctured Riemann surfaces Σ_1 and Σ_2. We single out a puncture on each surface, labelled puncture one and puncture two, and having local coordinates w_1 and w_2, respectively. Let $\Sigma(q)$ denote the surface produced by sewing together these surfaces with sewing parameter q. The surface state corresponding to the sewn surface is given by,

$$\langle\Sigma(q)| = \langle\Sigma_1;w_1\,|\langle\Sigma_2;w_2|q^{L_0^{(1)}}\bar{q}^{\overline{L}_0^{(1)}}|R_{12}\rangle\,, \quad \Sigma(q)\equiv\Sigma_1\cup_q\,\Sigma_2 : w_1w_2=q\ . \tag{3.1}$$

Note that the second state space could equally well be used for the Virasoro operators. This follows from the exchange symmetry of $|R_{12}\rangle$.

Consider now the canonical two punctured sphere $z_1(z)=z$, $z_2(z)=1/z$, with an additional puncture at $z=1$ whose local coordinate w will be defined as,

$$w=\frac{1}{2\pi i}\ln z\,. \tag{3.2}$$

This three punctured sphere will be denoted by R_{123}, and the corresponding surface state by $\langle R_{123}|$. The sphere is described in the w-plane as the cylindrical region determined by the identification: $w\sim w+1$.

Suppose we now sew the coordinates z_1 and z_2 via the identification $z_1z_2=q$, where $q\equiv\exp(2\pi i\tau)$. In terms of the z uniformizer this means identifying according to $z\sim z\,\exp(2\pi i\tau)$, which in w coordinates reads $w\sim w+\tau$. It follows from the identifications that we have obtained a torus with Teichmüller parameter τ. The surface state $\langle\Sigma(\tau);w|$ describing this once-punctured torus may therefore be written as,

$$\langle\Sigma(\tau);w| = \langle R_{123}|q^{L_0^{(1)}}\bar{q}^{\overline{L}_0^{(1)}}|R_{12}\rangle\,. \tag{3.3}$$

This is the once-punctured torus $w\sim w+1$, $w\sim w+\tau$, with local coordinate w at the puncture $w=0$.

We now consider describing a general family of once-punctured tori. The local coordinate at the puncture will depend on the modulus of the torus. Every torus will be explicitly realized in the w-plane via the identifications $w \sim w+1$ and $w \sim w+\tau$, and the puncture will be at $w=0$. As we change the modulus, the local coordinate at this puncture is conveniently described by a function showing how the unit coordinate disk $|\xi| \leq 1$ embeds into the torus,

$$w = h_{\tau,\bar{\tau}}(\xi) = a(\tau,\bar{\tau})\,\xi + \tfrac{1}{2} b(\tau,\bar{\tau})\,\xi^2 + \cdots \tag{3.4}$$

Note that the function need not depend holomorphically on the modulus τ. The above equation may be inverted to give the local coordinate as a function of the uniformizer w of the torus, $\xi = \xi(\tau,\bar{\tau},w)$. For brevity, we will write $\xi = \xi_\tau(w)$, and the w dependence will be left implicit in some cases.

Following (3.3), the surface state for a torus of modulus τ having coordinate ξ_τ at the puncture is given by,

$$\langle \Sigma(\tau);\xi_\tau| = \langle R_{123}^{\xi_\tau}\,|q^{L_0^{(1)}}\,\bar{q}^{\overline{L}_0^{(1)}}\,|R_{12}\rangle\,, \tag{3.5}$$

where the three punctured sphere $R_{123}^{\xi_\tau}$ is the canonical two-punctured sphere R_{12} equipped with an extra puncture at $z=1$:

$$R_{123}^{\xi_\tau}: \quad z_1(z) = z, \quad z_2(z) = 1/z\,, \quad z_3(z) = \xi_\tau(w(z)\,)\,. \tag{3.6}$$

Here w is the coordinate defined in (3.2). The choice of coordinate around the third puncture of the sphere ensures that the torus will have the desired local coordinate $\xi_\tau(w)$ at the puncture.

3.2. Forms on the moduli space of punctured tori

Our aim here is to write the canonical string forms on the moduli space $\mathcal{P}_{1,1}$ of once-punctured tori. The general case becomes clear once we write forms on the subspace of $\mathcal{P}_{1,1}$ parametrized by (3.4) which defines the local coordinate at the puncture as a function of the modulus of the torus. This is the case because the most general tangent to $\mathcal{P}_{1,1}$ is a tangent that changes the modulus of the torus and adjusts the local coordinate at the puncture.

We consider a torus of modulus τ and write the one-form and two-forms as

$$\begin{aligned}
\langle \Omega^{[1]1,1}(\tau)| &= \mathcal{N}^{-1}\,\langle R_{123}^{\xi_\tau}|q^{L_0^{(1)}}\,\bar{q}^{\overline{L}_0^{(1)}}\,|R_{12}\rangle\,\Big[d\tau b^{(3)}\Big(\frac{\partial}{\partial\tau}\Big) + d\bar{\tau}\,b^{(3)}\Big(\frac{\partial}{\partial\bar{\tau}}\Big)\Big]\,,\\
\langle \Omega^{[2]1,1}(\tau)| &= \mathcal{N}^{-1}\,\langle R_{123}^{\xi_\tau}|q^{L_0^{(1)}}\,\bar{q}^{\overline{L}_0^{(1)}}\,|R_{12}\rangle\;d\tau\wedge d\bar{\tau}\;b^{(3)}\Big(\frac{\partial}{\partial\tau}\Big)\,b^{(3)}\Big(\frac{\partial}{\partial\bar{\tau}}\Big)\,,
\end{aligned} \tag{3.7}$$

where $\mathcal{N} = -2\pi i$ is a normalization factor, and the surface state representing the once punctured torus was built by sewing. As usual, the antighost insertions are located at the free

puncture. The vectors $\partial/\partial\tau$ and $\partial/\partial\bar{\tau}$ denote the Schiffer vectors representing the deformation of the modulus together with the corresponding deformation of the local coordinates. Since surface states for punctured spheres are well-known [6,2], the construction of the above forms only requires the explicit expressions for the Schiffer vectors for the family of tori described in (3.4), and the prescription to build the corresponding antighost insertions.

Let us first consider the case where regardless of the modulus of the torus, the local coordinate at $w = 0$ is always taken to be the coordinate w. In this case we say, with a slight abuse of language, that the coordinate does not change as we change the modulus. The Schiffer vectors must not deform the local coordinate at the free puncture; in terms of the three-punctured sphere, the torus deformation only entails a change of sewing parameter for the sewing of the first two punctures. It is therefore convenient to place the antighost insertions on an interior puncture, say puncture one. The two-form then reads

$$\langle\Omega^{[2]1,1}(\tau)| = \mathcal{N}^{-1}\,\langle R_{123}|q^{L_0^{(1)}}\,\bar{q}^{\overline{L}_0^{(1)}}\,b^{(1)}\Big(\frac{\partial}{\partial\tau}\Big)b^{(1)}\Big(\frac{\partial}{\partial\bar{\tau}}\Big)|R_{12}\rangle\,d\tau\wedge d\bar{\tau}\ . \tag{3.8}$$

In this presentation, it is a standard calculation to show that,

$$b\Big(\frac{\partial}{\partial\tau}\Big) = -2\pi i b_0\,, \qquad b\Big(\frac{\partial}{\partial\bar{\tau}}\Big) = 2\pi i\bar{b}_0\,, \tag{3.9}$$

which can now be used to write

$$\langle\Omega^{[2]1,1}(\tau)| = -\mathcal{N}\cdot\langle R_{123}|q^{L_0^{(1)}}\,\bar{q}^{\overline{L}_0^{(1)}}\,b_0^{(1)}\bar{b}_0^{(1)}|R_{12}\rangle\,d\tau\wedge d\bar{\tau}\ . \tag{3.10}$$

<u>The Schiffer vectors on a torus</u>. We can now discuss Schiffer vectors $v(w)$ defined on the neighborhood of the puncture in the once-punctured torus.⋆ In the brief analysis which follows, we shall consider the possible vectors according to their behavior near the puncture at $w = 0$. Let us begin by considering those vectors fields which vanish at the puncture,

$$v(w) = w^n + \mathcal{O}(w^{n+1})\,, \quad n \geq 1\,. \tag{3.11}$$

Such objects do not extend holomorphically throughout the torus since no bounded elliptic function exists. Since they extend all the way inwards to $w = 0$, where they vanish, their effect is to change the local coordinate at the puncture. Corresponding to (3.4), the explicit form of the Schiffer vectors that change the local coordinates as τ changes, can be read from eqn.(6.10) of Ref.[2]

$$v_\tau(w) = -\frac{\partial h}{\partial\tau}(\xi(w))\,, \quad v_{\bar{\tau}}(w) = -\frac{\partial h}{\partial\bar{\tau}}(\xi(w))\,. \tag{3.12}$$

These vectors are clearly of the type indicated in (3.11).

⋆ For a pedagogic discussion of general properties of Schiffer vectors and Schiffer variations see Ref.[7].

The constant vector $v(w) = 1$ is well defined throughout the torus and therefore changes neither the modulus nor the local coordinate at the puncture. The case of a vector with first order pole is more interesting:

$$v(w) = \frac{1}{w} + \mathcal{O}(w^0) \ . \tag{3.13}$$

This vector cannot be extended analytically throughout the torus as no elliptic function with a single first order pole exists. Such a vector field, which neither extends inwards nor outwards, must change the modulus. We are especially interested in finding the vector field which changes the modulus without a corresponding change in local coordinate at the puncture. To this end, it is useful to consider the logarithmic derivative of the Jacobi theta function $\theta_1(w|\tau)$,

$$u(w\,|\tau) \equiv \frac{\theta_1'(w\,|\tau)}{\theta_1(w\,|\tau)} = \frac{1}{w} + \cdots \ , \tag{3.14}$$

where the prime denotes differentiation with respect to the first argument. (We follow the conventions of Ref.[8]; note however that there $q = e^{i\pi\tau}$). This logarithmic derivative has simple properties under translations,

$$u(w+\pi\,|\tau) = u(w\,|\tau)\,, \qquad u(w+\pi\tau\,|\tau) = -2i + u(w\,|\tau)\,. \tag{3.15}$$

For our purposes, it is convenient to choose the particular Schiffer vector,

$$v_o(w) = \frac{i}{2}\,u(\pi w\,|\tau)\,, \tag{3.16}$$

which, by virtue of (3.15) satisfies,

$$v_o(w+1) = v_o(w)\,, \quad v_o(w+\tau) = 1 + v_o(w)\,. \tag{3.17}$$

As will be seen later, this choice of Schiffer vector is tailored precisely to change the modulus without changing the local coordinate at the puncture (again, in the sense that its dependence on the uniformizer is unaltered). Since θ-functions are entire, the only pole of v_o is at $w = 0$.

To conclude our analysis, consider vectors of the form,

$$v(w) = \frac{1}{w^n} + \mathcal{O}(w^{1-n})\,, \quad n \geq 2\,. \tag{3.18}$$

Vectors with such leading behavior can always be reduced to one of the cases already discussed by subtracting a suitable linear combination of elliptic funtions. In particular the Weierstrass $\mathcal{P}$ function may be used to eliminate a second order pole, while its derivatives can be used to eliminate higher order poles.

The Antighost Insertions. We now describe the antighost insertions corresponding to the Schiffer vectors discussed above. We treat separately the contributions from vectors that change coordinates only and from vectors that only change moduli. Finally, the results are combined to give the expression of a general form on the moduli space of once-punctured tori.

The antighost insertions corresponding to the coordinate changing Schiffer vectors in (3.12) are given by

$$\begin{aligned} b(v_\tau) &= \oint \frac{dw}{2\pi i} b(w) v_\tau(w) + \oint \frac{d\bar{w}}{2\pi i}\, \bar{b}(\bar{w})\, \overline{v_{\bar{\tau}}(w)}\,, \\ b(v_{\bar{\tau}}) &= \oint \frac{dw}{2\pi i} b(w) v_{\bar{\tau}}(w) + \oint \frac{d\bar{w}}{2\pi i} \bar{b}(\bar{w}) \overline{v_\tau(w)}\,, \end{aligned} \tag{3.19}$$

where overbar on a vector denotes complex conjugation, and we integrate using $\oint dw/2\pi i w = \oint d\bar{w}/2\pi i \bar{w} = 1$. In these equations $b(v_\tau)$ and $b(v_{\bar{\tau}})$ are simply the portions of $b(\partial/\partial\tau)$ and of $b(\partial/\partial\bar{\tau})$ representing the coordinate changes due to the change in modulus. The explicit oscillator form of the antighost insertions follows readily from the above equations once the Schiffer vectors in (3.12) are written as power series in w. Note that the insertions are acting on the external puncture, but use the w uniformizer. This is convenient when it comes to re-expressing them in terms of the first and second state spaces, the spaces that are traced over. Since b is primary, (3.19) is coordinate independent and the insertions can be expressed in terms of oscillators $b^{(\xi)}$ by using the Schiffer vectors referred to the ξ coordinates. This may be useful since the external state is written in terms of such oscillators.

Let us now consider the antighost insertion corresponding to a change in modulus only. The vector field we need is precisely the vector $v_o(w)$ introduced in (3.16). If we define

$$b(v_o) = \oint \frac{dw}{2\pi i} b(w) v_o(w)\,, \quad b(\overline{v_o}) = \oint \frac{d\bar{w}}{2\pi i} \bar{b}(\bar{w}) \overline{v_o(w)}, \tag{3.20}$$

then, the correctness of our claim requires that

$$\langle R^{\xi_\tau}_{123}|\, q^{L_0^{(1)}}\, \bar{q}^{\overline{L}_0^{(1)}}\, |R_{12}\rangle\, b(v_o) = \langle R^{\xi_\tau}_{123}|\, (-2\pi i\, b_0^{(1)})\, q^{L_0^{(1)}}\, \bar{q}^{\overline{L}_0^{(1)}}\, |R_{12}\rangle\,. \tag{3.21}$$

This equation relates an antighost insertion on the puncture to an antighost operator inside the trace and by virtue of (3.9) it justifies the assertion that the antighost insertion $b(v_o)$ describes a change of modulus only, without affecting the local coordinate at the puncture. The proof of this identity is given in the appendix. In a similar manner,

$$\langle R^{\xi_\tau}_{123}|\, q^{L_0^{(1)}}\, \bar{q}^{\overline{L}_0^{(1)}}\, |R_{12}\rangle\, b(\overline{v_o}) = \langle R^{\xi_\tau}_{123}|\, (2\pi i\, \bar{b}_0^{(1)})\, q^{L_0^{(1)}}\, \bar{q}^{\overline{L}_0^{(1)}}\, |R_{12}\rangle\,. \tag{3.22}$$

Our results are now complete. The antighost insertions representing *both* changes of moduli

and local coordinates are given as

$$b\left(\frac{\partial}{\partial\tau}\right)=b(v_o)+b(v_\tau)\,,\qquad b\left(\frac{\partial}{\partial\bar\tau}\right)=b(\overline{v_o})+b(v_{\bar\tau})\,. \tag{3.23}$$

These expressions can now be substituted back in (3.7) and give the explicit expressions for the canonical string forms on the moduli space of once-punctured tori.

4. The Case of Genus One

In this section, we apply the earlier results to prove the ghost-dilaton theorem at genus one. According to [1], the theorem is proven if the following equation holds,

$$f_D(\underline{\mathcal{V}}_{1,1})-f_\chi(\Delta\underline{\mathcal{V}}_{0,3})=\kappa\frac{d}{d\kappa}\left(S_{1,0}+\frac{1}{2}\ln\rho\right)\,. \tag{4.1}$$

The expression in parenthesis in the right hand side corresponds to the elementary contribution to the one-loop free energy appearing in the string action. It is expected to be independent of the string coupling κ, so ideally one would hope that the left hand side of the equation vanishes. It does. In what follows, we verify that in fact both terms appearing in this left hand side vanish independently.

Preliminary remark. We shall consider first $f_\chi(\Delta\underline{\mathcal{V}}_{0,3})$, with $|\chi\rangle=-c_0^-|0\rangle$,

$$f_\chi(\Delta\mathcal{V}_{0,3})=\int_{\Delta\mathcal{V}_{0,3}}\frac{d\theta}{2\pi}\,\langle V_{123}|b_0^{-(1)}e^{i\theta L_0^{-(1)}}|R_{12}\rangle c_0^{-(3)}|0\rangle_3\,. \tag{4.2}$$

The geometrical interpretation is that we take our choice of three-string vertex (denoted here by $\langle V_{123}|$), and twist-sew two of the punctures, inserting the state $|\chi\rangle$ at the third puncture. It happens that certain simplifications occur in the special case where the three-string vertex is chosen to be the Witten vertex $\langle W_{123}|$. This example gives an insight into how the expression may be evaluated for the general case.

The $\langle W_{123}|$ vertex satisfies the conservation law: $\langle W_{123}|(c_0^{(1)}+c_0^{(2)}+c_0^{(3)})=0$, which allows us to replace the $c_0^{-(3)}$ in (4.2) by the factor $(c_0^{-(1)}+c_0^{-(2)})$ immediately to the right of the surface state. This factor annihilates the reflector $|R_{12}\rangle$, and all that remains is the term picked up by anticommutation with $b_0^{-(1)}$,

$$f_\chi(\Delta\mathcal{V}_{0,3})=\int_{\Delta\mathcal{V}_{0,3}}\frac{d\theta}{2\pi}\,\langle W_{123}|e^{i\theta L_0^{-(1)}}|R_{12}\rangle\,|0\rangle_3\,. \tag{4.3}$$

Let us consider the integrand of this expression. We have a three-punctured sphere with two punctures sewn together, and a state (in this case the vacuum) inserted in the last puncture.

Such an object is a once-punctured torus of some modulus $\tau(\theta)$. The calculation of τ as a function of θ is in general nontrivial, but will not be necessary for our arguments. Referring back to our discussion of §3.1, any once-punctured torus with arbitrary modulus τ can be made by sewing together the punctures of $R^{\xi_\tau}_{123}$ with an appropriately chosen sewing parameter, and local coordinate ξ_τ. We may therefore rewrite (4.3) as,

$$f_\chi(\Delta\mathcal{V}_{0,3}) = \int\limits_{\Delta\mathcal{V}_{0,3}} \frac{d\theta}{2\pi} \langle R^{\xi_\tau}_{123}| q^{L_0^{(1)}} \overline{q}^{\overline{L}_0^{(1)}} |R_{12}\rangle \, |0\rangle_3 \; = \int\limits_{\Delta\mathcal{V}_{0,3}} \frac{d\theta}{2\pi} \langle R_{12}| q^{L_0^{(1)}} \overline{q}^{\overline{L}_0^{(1)}} |R_{12}\rangle \,, \tag{4.4}$$

where $q \equiv e^{2\pi i\tau(\theta)}$. It becomes clear that we are actually dealing with the supertrace of the operator $q^{L_0}\overline{q}^{\overline{L}_0}$ ⋆.

The supertrace of any operator vanishes unless the operator explicitly contains the factor $b_0c_0\bar{b}_0\bar{c}_0$. The necessity of the factor b_0c_0 follows by considering the holomorphic sector. The state space splits into two sectors, identical except for the fact that they are built upon the vacua $|0\rangle$ and $c_0|0\rangle$ respectively. The different fermion numbers of these two sectors means that their contributions to the supertrace cancel. It is therefore necessary to project out one of these sectors while still conserving ghost number, the only operator suitable for this being b_0c_0. Similarly, the antiholomorphic sector requires the factor $\bar{b}_0\bar{c}_0$.

Since neither L_0 nor $\overline{L}_0$ contain ghost zero modes, the integrand in (4.4) vanishes identically. Note that such simple arguments could not be made for (4.3). Upon deletion of the third puncture, the leftover two-punctured sphere does not coincide with R_{12} and the absence of ghost zero modes is not manifest.

<u>The general case.</u> We now consider to the case of the general three-string vertex. Since $f_\chi(\Delta\mathcal{V}_{0,3})$ is simply an integral of the canonical string one-form along a one-parameter subspace of $\mathcal{P}_{1,1}$, the considerations of the previous section indicate that it can be written as

$$f_\chi(\Delta\mathcal{V}_{0,3}) = \mathcal{N}^{-1} \int\limits_{\Delta\mathcal{V}_{0,3}} \langle R^{\xi_\tau}_{123}| q^{L_0^{(1)}} \overline{q}^{\overline{L}_0^{(1)}} |R_{12}\rangle \Big[d\tau\, b^{(3)}\Big(\frac{\partial}{\partial\tau}\Big) + d\overline{\tau}\, b^{(3)}\Big(\frac{\partial}{\partial\overline{\tau}}\Big)\Big] |\chi\rangle_3 \,, \tag{4.5}$$

Here the tori are built by gluing two punctures of the τ-dependent $R^{\xi_\tau}_{123}$ spheres, where the coordinates at the third puncture are determined by the desired coordinates on the punctured tori $\Delta\mathcal{V}_{0,3}$. The antighost insertions represent both changes of moduli, and changes in the local coordinates as outlined in detail in the previous section (see (3.23)).

An important property of the $R^{\xi_\tau}_{123}$ sphere is that the local coordinates at punctures one and two reach puncture three. This means that we can, by way of conformal transformations, map any operator acting on the third state space to an operator acting on the first or on the

⋆ One can readily show that $\langle R_{12}|A^{(1)}|R_{12}\rangle = \sum_s (-)^s \langle \Phi^s|A|\Phi_s\rangle \equiv \mathrm{str}(A)$

second state space. Such mapping couples neither left-movers with right-movers nor ghosts with antighosts. Exploiting this fact to our advantage, we use conformal mapping to transfer all oscillators in the third puncture to the first puncture. Once again, the vacuum state deletes the special puncture of $\langle R_{123}^{\xi_\tau}|$, and what remains is an expression of the form,

$$f_\chi(\Delta\mathcal{V}_{0,3}) \sim \int\limits_{\Delta\mathcal{V}_{0,3}} \langle R_{12}| \Big(d\tau (b^{(1)} + \overline{b}^{(1)}) + d\overline{\tau} (b^{(1)} + \overline{b}^{(1)}) \Big) (c^{(1)} - \overline{c}^{(1)}) q^{L_0^{(1)}} \overline{q}^{\overline{L}_0^{(1)}} |R_{12}\rangle \, , \quad (4.6)$$

where each ghost or antighost term indicated schematically above, is in general a sum of many modes. For our present purposes we only need to know the holomorphic or antiholomorphic characted of a given term. This is displayed by the absence or presence of an overbar. It is now quite clear that at most two of the required four ghost zero modes may be present in any one term. The integrand therefore vanishes identically.

We can now apply the same ideas to $f_D(\underline{\mathcal{V}}_{1,1})$

$$\begin{aligned} f_D(\underline{\mathcal{V}}_{1,1}) &= \mathcal{N}^{-1} \int\limits_{\underline{\mathcal{V}}_{1,1}} d\tau \wedge d\overline{\tau} \, \langle R_{123}^{\xi_\tau} | q^{L_0^{(1)}} \overline{q}^{\overline{L}_0^{(1)}} |R_{12}\rangle \, b^{(3)}\Big(\frac{\partial}{\partial\tau}\Big) \, b^{(3)}\Big(\frac{\partial}{\partial\overline{\tau}}\Big) |D\rangle_3 \\ &\sim \int\limits_{\underline{\mathcal{V}}_{1,1}} d\tau \wedge d\overline{\tau} \, \langle R_{12}| (b^{(1)} + \overline{b}^{(1)}) (b^{(1)} + \overline{b}^{(1)}) (c^{(1)} c^{(1)} - \overline{c}^{(1)} \overline{c}^{(1)}) q^{L_0^{(1)}} \overline{q}^{\overline{L}_0^{(1)}} |R_{12}\rangle \, , \end{aligned} \quad (4.7)$$

where once again we used conformal mapping to move all oscillators to the first puncture (recall $|D\rangle = (c_1 c_{-1} - \bar{c}_1 \bar{c}_{-1})|0\rangle$). This time the expression vanishes due to the absence of the combination $c_0 \overline{c}_0$. We have thereby shown that the left hand side of (4.1) vanishes. This concludes our proof of the ghost-dilaton theorem for the field independent terms in the string action.

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APPENDIX

We consider here the proof of (3.21). The starting point is

$$\langle R_{123}^{\xi_\tau}| \oint_{\mathcal{C}_0} \frac{dw}{2\pi i} b(w) v_o(w)\, q^{L_0} \bar{q}^{\overline{L}_0} |R_{12}\rangle\,, \tag{A.1}$$

where the integral is around the point $w = 0$. Note that the antighost insertion is written in terms of the w-coordinate at the puncture. This expression is viewed as an integral on the sphere $R_{123}^{\xi_\tau}$. (See Fig.1).

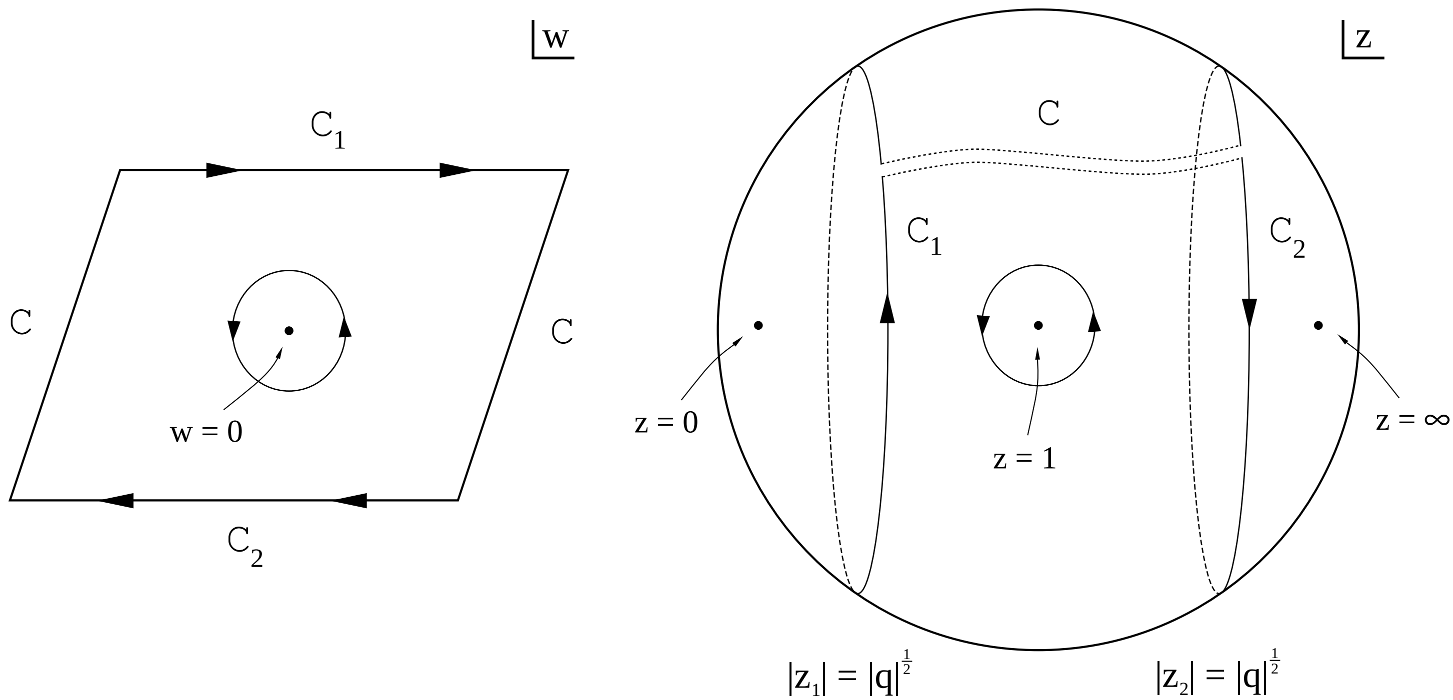

Figure 1. The torus of modulus τ built by sewing two punctures in the sphere $R_{123}^{\xi_\tau}$.

The periodicity $v_o(w) = v_o(w{+}1)$ in (3.17) implies that the vector field v_o is well defined on the annulus bounded by the curves $\mathcal{C}_1$ and $\mathcal{C}_2$, corresponding to $|z_1| = |q|^{1/2}$ and $|z_2| = |q|^{1/2}$, respectively. Since v_o is analytic on the annulus minus the puncture, it follows by contour deformation that the integral around $w = 0$ is equal to the sum of two integrals, one over $\mathcal{C}_1$ and the other over $\mathcal{C}_2$:

$$\langle R_{123}^{\xi_\tau}| \Big[-\oint_{\mathcal{C}_1} \frac{dw}{2\pi i} b(w) v_o(w) \,-\oint_{\mathcal{C}_2} \frac{dw}{2\pi i} b(w) v_o(w) \Big]\, q^{L_0^{(1)}} \bar{q}^{\overline{L}_0^{(1)}} |R_{12}\rangle\,. \tag{A.2}$$

Consider the antighost field $b(w)$ in the second integral and express it in terms of z_2 coordinates,

$$b(w)\cdots q^{L_0^{(1)}}\bar{q}^{\overline{L}_0^{(1)}}|R_{12}\rangle = \left(\frac{dz_2}{dw}\right)^2 b(z_2)\cdots q^{L_0^{(1)}}\bar{q}^{\overline{L}_0^{(1)}}|R_{12}\rangle\,. \tag{A.3}$$

The antighost, living in the second state space, can be taken all the way to the reflector $|R_{12}\rangle$ at which point it can be re-expressed in terms of the coordinate z_1,

$$b(w)\cdots q^{L_0^{(1)}}\bar{q}^{\overline{L}_0^{(1)}}|R_{12}\rangle = \left(\frac{dz_1}{dw}\right)^2 \cdots q^{L_0^{(1)}}\bar{q}^{\overline{L}_0^{(1)}} b(z_1)|R_{12}\rangle\,. \tag{A.4}$$

In order to bring the antighost back to the left we use $q^{L_0}b(z) = q^2 b(qz)$,

$$b(w)\cdots q^{L_0^{(1)}}\bar{q}^{\overline{L}_0^{(1)}}|R_{12}\rangle = \left(\frac{dz_1}{dw}\right)^2 q^2 b(qz_1)\,\cdots q^{L_0^{(1)}}\bar{q}^{\overline{L}_0^{(1)}}|R_{12}\rangle\,, \tag{A.5}$$

and expressing the antighost back in w coordinates we obtain,

$$\left(\frac{dz_1}{dw}\right)^2 q^2 \left(\frac{dw}{dz_1}\right)^2_{qz_1} b(w+\tau)\,\cdots q^{L_0^{(1)}}\bar{q}^{\overline{L}_0^{(1)}}|R_{12}\rangle\,. \tag{A.6}$$

The factor in front of the antighost field is precisely unity. Returning to (A.2), we have,

$$\langle R_{123}^{\xi_\tau}|\Big[-\oint_{\mathcal{C}_1}\frac{dw}{2\pi i}b(w)v_o(w)\,-\oint_{\mathcal{C}_2}\frac{dw}{2\pi i}b(w+\tau)v_o(w)\Big]\,q^{L_0^{(1)}}\bar{q}^{\overline{L}_0^{(1)}}|R_{12}\rangle\,. \tag{A.7}$$

Using $v_o(w+\tau) = 1 + v_o(w)$, and a change of variable to move the contour from $\mathcal{C}_2$ to $-\mathcal{C}_1$,

$$-\oint_{\mathcal{C}_2}\frac{dw}{2\pi i}\,b(w+\tau)v_o(w) = -\oint_{\mathcal{C}_2}\frac{dw}{2\pi i}\,b(w+\tau)\,[v_o(w+\tau)-1\,] = \oint_{\mathcal{C}_1}\frac{dw}{2\pi i}\,b(w)\,[v_o(w)-1\,]\,. \tag{A.8}$$

The terms containing $v_o(w)$ in (A.7) then cancel, leaving only,

$$\langle R_{123}^{\xi_\tau}|\Big[-\oint_{\mathcal{C}_1}\frac{dw}{2\pi i}b(w)\Big]\,q^{L_0^{(1)}}\bar{q}^{\overline{L}_0^{(1)}}|R_{12}\rangle\,. \tag{A.9}$$

This integral is evaluated in the z_1 coordinate and gives simply $(2\pi i)b_0^{(1)}$. We therefore have

$$\langle R_{123}^{\xi_\tau}|\,q^{L_0^{(1)}}\bar{q}^{\overline{L}_0^{(1)}}|R_{12}\rangle\,\oint_{\mathcal{C}_0}\frac{dw}{2\pi i}b(w)v_o(w) = \langle R_{123}^{\xi_\tau}|\,(-2\pi i\,b_0^{(1)})\,q^{L_0^{(1)}}\bar{q}^{\overline{L}_0^{(1)}}|R_{12}\rangle\,. \tag{A.10}$$

This was the desired result.

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